We study in this paper magnetic properties of a system of quantum Heisenberg spins interacting with each other via a ferromagnetic exchange interaction $J$ and an in-plane Dzyaloshinskii-Moriya interaction $D$. The non-collinear ground state due to the competition between $J$ and $D$ is determined. We employ a self-consistent Green function theory to calculate the spin-wave spectrum and the layer magnetizations at finite $T$ in two and three dimensions as well as in a thin film with surface effects. Analytical details and the validity of the method are shown and discussed.

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I. INTRODUCTION

The Dzyaloshinskii-Moriya (DM) interaction was proposed to explain the weak ferromagnetism which was observed in antiferromagnetic Mn compounds. The phenomenological Landau-Ginzburg model introduced by I. Dzyaloshinskii [1] was microscopically derived by T. Moriya [2]. The interaction between two spins $S_i$ and $S_j$ is written as

$$D_{i,j} \cdot S_i \wedge S_j$$

where $D_{i,j}$ is a vector which results from the displacement of non magnetic ions located between $S_i$ and $S_j$, for example in Mn-O-Mn bonds. The direction of $D_{i,j}$ depends on the symmetry of the displacement [2]. The DM interaction is antisymmetric with respect to the inversion symmetry.

There has been a large number of investigations on the effect of the DM interaction in various materials, both experimentally and theoretically for weak ferromagnetism which was observed in antiferromagnetic Mn compounds. The symmetry nearest-neighbor (NN) and next-nearest neighbor (NNN) interactions [20, 21], the DM interaction, as said above, is antisymmetric. This gives rise to a non trivial SW behavior as will be shown below. Note that there has been a number of works dealing with the SW properties in DM systems [22–26].

This paper is organized as follows. Section II is devoted to the description of the model and the determination of the GS. Section III shows the formulation of our self-consistent Green’s function (GF) method. Section IV shows results on the SW spectrum and the magnetization in two dimensions (2D) and three dimensions (3D). The case of thin films with free surfaces is shown in section V where layer magnetizations at finite temperature ($T$) and the thickness effect are presented. Concluding remarks are given in section VI.

II. MODEL AND GROUND STATE

We consider a thin film of simple cubic (SC) lattice of $N$ layers stacked in the $y$ direction perpendicular to the film surface. For the reason which is shown below, we choose the film surface as a $xz$ plane. The Hamiltonian is given by
\[ H = H_e + H_{DM} \]  \hspace{1cm} (2)

\[ H_e = -\sum_{\langle i,j \rangle} J_{ij} S_i \cdot S_j \]  \hspace{1cm} (3)

\[ H_{DM} = \sum_{\langle i,j \rangle} D_{i,j} \cdot S_i \wedge S_j \]  \hspace{1cm} (4)

where \( J_{ij} \) and \( D_{i,j} \) are the exchange and DM interactions, respectively, between two Heisenberg spins \( S_i \) and \( S_j \) of magnitude \( S = 1/2 \) occupying the lattice sites \( i \) and \( j \).

For simplicity, let us consider the case where the in-plane and inter-plane exchange interactions between NN are both ferromagnetic and denoted by \( J_i \) and \( J_\perp \), respectively. The DM interaction is supposed to be between NN in the plane with a constant \( D \). Due to the competition between the exchange \( J \) term which favors the collinear configuration, and the DM term which favors the perpendicular one, we expect that the spin \( S_i \) makes an angle \( \theta_{i,j} \) with its neighbor \( S_j \). Therefore, the quantization axis of \( S_i \) is not the same as that of \( S_j \). Let us call \( \hat{\xi}_i \) the quantization axis of \( S_i \) and \( \hat{\xi}_j \) its perpendicular axis in the \( xz \) plane. The third axis \( \hat{\eta}_i \), perpendicular to the film surface, is chosen in such a way to make \( (\hat{\xi}_i, \hat{\eta}_i, \hat{\zeta}_i) \) an orthogonal direct frame. Writing \( S_i \) and \( S_j \) in their respective local coordinates, one has

\[ S_i = S_i^x \hat{\xi}_i + S_i^y \hat{\eta}_i + S_i^z \hat{\zeta}_i \]  \hspace{1cm} (5)

\[ S_j = S_j^x \hat{\xi}_j + S_j^y \hat{\eta}_j + S_j^z \hat{\zeta}_j \]  \hspace{1cm} (6)

We choose the vector \( D_{i,j} \) perpendicular to the \( xz \) plane, namely

\[ D_{i,j} = D e_{i,j} \hat{\eta}_i \]  \hspace{1cm} (7)

where \( e_{i,j} = +1 \) (-1) if \( j > i \) (\( j < i \)) for NN on the \( \hat{\xi}_i \) or \( \hat{\zeta}_i \) axis. Note that \( e_{j,i} = -e_{i,j} \).

To determine the GS, the easiest way is to use the steepest descent method: we calculate the local field acting on each spin from its neighbors and we align the spin in its local-field direction to minimize its energy. Repeating this for all spins and iterating many times until the convergence is reached with a desired precision (usually at the 6-th digit, namely at \( \simeq 10^{-6} \) per cents), we obtain the lowest energy state of the system (see Ref. 27). Note that we have used several thousands of different initial conditions to check the convergence to a single GS for each set of parameters. Choosing \( D_{i,j} \) lying perpendicular to the spin plane (i.e. \( xz \) plane) as indicated in Eq. (7), we determine the GS as a function of \( D \). An example is shown in Fig. 1 for \( \theta = \pi/6 \) \((D = -0.577)\) with \( J_i = J_\perp = 1 \). We see that each spin has the same angle with its four NN in the plane (angle between NN in adjacent planes is zero). Let us show the relation between \( \theta \) and \( J_\perp \): the energy of the spin \( S_i \), written as

\[ E_i = -4J_i S^2 \cos \theta - 2J_\perp S^2 + 4DS^2 \sin \theta \]  \hspace{1cm} (8)

where \( \theta = |\theta_{i,j}| \) and care has been taken on the signs of \( \sin \theta_{i,j} \) and \( e_{i,j} \) when counting NN, namely two opposite NN have opposite signs. The minimization of \( E_i \) yields

\[ \frac{dE_i}{d\theta} = 0 \Rightarrow -\frac{D}{J_i} = \tan \theta \Rightarrow \theta = \arctan\left(-\frac{D}{J_i}\right) \]  \hspace{1cm} (9)

The value of \( \theta \) for a given \( \frac{D}{J_i} \) is precisely what obtained by the steepest descent method.

In the present model, the DM interaction is supposed in the plane, so in the GS the angle between in-plane NN is not zero. We show in Fig. 1 the relative orientation of the two NN spins in the plane.

Note that the perpendicular axes \( \hat{\eta}_i \) and \( \hat{\eta}_j \) coincide.

Now, expressing the local frame of \( S_j \) in the local frame of \( S_i \), we have

\[ \hat{\zeta}_j = \cos \theta_{i,j} \hat{\xi}_j + \sin \theta_{i,j} \hat{\xi}_i \]  \hspace{1cm} (10)

\[ \hat{\xi}_j = -\sin \theta_{i,j} \hat{\xi}_i + \cos \theta_{i,j} \hat{\xi}_i \]  \hspace{1cm} (11)

\[ \hat{\eta}_j = \hat{\eta}_i \]  \hspace{1cm} (12)
us briefly recall here the principal steps of calculation and give the results for the present model. Expressing the Hamiltonian in the local coordinates, we obtain

\[ H = - \sum_{<i,j>} J_{i,j} \left\{ \frac{1}{4} (\cos \theta_{i,j} - 1) \left( S^+_i S^-_j + S^-_i S^+_j \right) + \frac{1}{4} (\cos \theta_{i,j} + 1) \left( S^+_i S^-_j + S^-_i S^+_j \right) + \frac{1}{2} \sin \theta_{i,j} \left( S^+_i + S^-_i \right) S^-_j - \frac{1}{2} \sin \theta_{i,j} S^+_i \left( S^+_j + S^-_j \right) + \cos \theta_{i,j} S^i_z S^j_z \right\} \]

(16)

As said in the previous section, the spins lie in the \(xz\) planes, each on its quantization local \(z\) axis (Fig. 2).

Note that unlike the sinus term of the DM Hamiltonian, Eq. (15), the sinus terms of \(H_a\), the 3rd line of Eq. (16), are zero when summed up on opposite NN (no \(e_{i,j}\) to compensate). The 3rd line disappears therefore in the following.

At this stage it is very important to note that the standard commutation relations between spin operators \(S^x\) and \(S^y\) are defined with \(z\) as the spin quantization axis. In non collinear spin configurations, calculations of SW spectrum using commutation relations without paying attention to this are wrong.

It is known that in two dimensions (2D) there is no long-range order at finite temperature \((T)\) for isotropic spin models with short-range interaction [34]. Thin films have small thickness, therefore to stabilize the ordering at finite \(T\) it is useful to add an anisotropic interaction. We use the following anisotropy between \(S_i\) and \(S_j\) which stabilizes the angle determined above between their local quantization axes \(S^i_z\) and \(S^j_z\):

\[ H_a = - \sum_{<i,j>} I_{i,j} S^i_z S^j_z \cos \theta_{i,j} \]

(17)

where \(I_{i,j}\) is supposed to be positive, small compared to \(J_{i,j}\), and limited to NN. Hereafter we take \(I_{i,j} = I_1\) for NN pair in the \(xz\) plane, for simplicity. As it turns out, this anisotropy helps stabilize the ordering at finite \(T\) in 2D as discussed. It helps also stabilize the SW spectrum at \(T = 0\) in the case of thin films but it is not necessary for 2D and 3D at \(T = 0\). The total Hamiltonian is finally given by

\[ H = H_a + H_{DM} + H_a \]

(18)

We define the following two double-time GF’s in the

\[ \Sigma_{ij} = \Sigma_{ij}^{(a)} + \Sigma_{ij}^{(b)} \]

(19)

where ...
real space

\[ G_{i,j}(t,t') = \langle\langle S^+_i(t); S^-_j(t') \rangle\rangle \]
\[ = -i\theta(t - t') \langle\langle [S^+_i(t), S^-_j(t')] \rangle\rangle \]  
\[ F_{i,j}(t,t') = \langle\langle S^-_i(t); S^-_j(t') \rangle\rangle \]
\[ = -i\theta(t - t') \langle\langle [S^-_i(t), S^-_j(t')] \rangle\rangle \]  

The equations of motion of these functions read

\[
\frac{i\hbar}{\partial t} G_{i,j}(t,t') = \langle\langle [H, S^+_i(t); S^-_j(t')] \rangle\rangle = -\langle\langle [\mathcal{H}, S^+_i(t)]; S^-_j(t') \rangle\rangle \]
\[
\frac{i\hbar}{\partial t} F_{i,j}(t,t') = \langle\langle [H, S^-_i(t); S^-_j(t')] \rangle\rangle = -\langle\langle [\mathcal{H}, S^-_i(t)]; S^-_j(t') \rangle\rangle \]

For the \( \mathcal{H}_e \) and \( \mathcal{H}_a \) parts, the above equations of motion generate terms such as \( \langle\langle S^+_i S^+_j; S^-_j \rangle\rangle \) and \( \langle\langle S^+_i S^-_j; S^-_j \rangle\rangle \). These functions can be approximated by using the Tyablikov decoupling to reduce to the above-defined \( G \) and \( F \) functions:

\[
\langle\langle S^+_i S^+_j; S^-_j \rangle\rangle \approx 2 \langle\langle S^+_i; S^-_j \rangle\rangle \]
\[
\langle\langle S^+_i S^-_j; S^-_j \rangle\rangle \approx 2 \langle\langle S^+_i; S^-_j \rangle\rangle \]  

The last expression is due to the fact that transverse SW motions \( S^\pm_i \) are zero with time. For the DM term, the commutation relations \([\mathcal{H}, S^\pm_i]\) give rise to the following term:

\[
D \sum_i \sin \theta [\langle\langle S^+_i; S^-_i \rangle\rangle \pm 2 \langle\langle S^+_i \rangle\rangle \langle\langle S^-_i \rangle\rangle] \]

which leads to the following type of GF's:

\[
\langle\langle S^+_i S^+_j; S^-_j \rangle\rangle \approx \langle\langle S^+_i; S^-_j \rangle\rangle \approx \langle\langle S^+_i; S^-_j \rangle\rangle \approx 0 \]

Note that we have replaced \( e_{i,j} \sin \theta_{i,j} \) by \( \sin \theta \) where \( \theta \) is positive. The above equation is related to \( G \) and \( F \) functions [see Eq. (24)]. The Tyablikov decoupling scheme neglects higher-order functions.

We now introduce the following in-plane Fourier transforms \( g_{n,n'} \) and \( f_{n,n'} \) of the \( G \) and \( F \) Green's functions:

\[
G_{i,j}(t,t',\omega) = \frac{1}{\Delta} \int_{\Delta} d\vec{k}_{zz} e^{-i\omega(t-t')} \times g_{n,n'}(\omega, \vec{k}_{zz}) e^{i\vec{k}_{zz}(\vec{R}_{i} - \vec{R}_{j})} \]
\[
F_{i,j}(t,t',\omega) = \frac{1}{\Delta} \int_{\Delta} d\vec{k}_{zz} e^{-i\omega(t-t')} \times f_{n,n'}(\omega, \vec{k}_{zz}) e^{i\vec{k}_{zz}(\vec{R}_{i} - \vec{R}_{j})} \]

where the integral is performed in the first \( \Delta \) Brillouin zone (\( \Delta \)) of surface \( \Delta \), \( \omega \) is the spin-wave frequency, \( n \) and \( n' \) are the indices of the layers along the \( c \) axis to which \( \vec{R}_{i} \) and \( \vec{R}_{j} \) belong (\( n = 1 \) being the surface layer, \( n = 2 \) the second layer and so on). We finally obtain the following matrix equation

\[
\mathbf{M}(E) \mathbf{h} = \mathbf{u}, \quad \mathbf{u} = \begin{pmatrix}
 g_{1,n'} \\
 f_{1,n'} \\
 \vdots \\
 g_{N,n'} \\
 f_{N,n'}
\end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix}
 2 \langle S^+_i \rangle \delta_{1,n'} \\
 0 \\
 \vdots \\
 2 \langle S^+_i \rangle \delta_{N,n'} \\
 0
\end{pmatrix}
\]

where \( E = \hbar \omega \) and \( \mathbf{M}(E) \) is given by

\[
\begin{pmatrix}
 E + A_1 & B_1 & C_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -B_1 & E - A_1 & 0 & -C_1 & 0 & 0 & 0 & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & 0 & C_n & 0 & E + A_n & B_n & C_n & 0 & 0 \\
 \vdots & 0 & 0 & -C_n & -B_n & E - A_n & 0 & -C_n & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & 0 & 0 & 0 & 0 & C_N & 0 & E + A_N & B_N \\
 0 & 0 & 0 & 0 & 0 & 0 & -C_N & -B_N & E - A_N & 0
\end{pmatrix}
\]

with

\[
A_n = -J_{\perp} [8 < S^+_i > \cos \theta(1 + d_n)] -4 < S^+_i > \gamma (\cos \theta + 1) - 2J_{\perp} (< S^+_i > + < S^+_i >) - 4D \sin \theta < S^+_i > \gamma + 8D \sin \theta < S^+_i > \\
B_n = 4J_{\perp} < S^+_i > \gamma (\cos \theta - 1) - 4D \sin \theta < S^+_i > \gamma \\
C_n = 2J_{\perp} < S^+_i > \gamma
\]

where \( n = 1, 2, ... , N \), \( d_n = I_{\perp}/J_{\perp} \), \( \gamma = (\cos k_x a + \cos k_z a)/2 \), \( k_x \) and \( k_z \) denote the wave-vector components.
in the $xz$ planes, $a$ the lattice constant. Note that (i) if
$n = 1$ (surface layer) then there are no $n - 1$ terms in the
matrix coefficients, (ii) if $n = N$ then there are no $n + 1$
terms. Besides, we have distinguished the in-plane NN
interaction $J_\parallel$ from the inter-plane NN one $J_\perp$.

In the case of a thin film, the SW eigenvalues at a given
wave vector $\vec{k} = (k_x, k_z)$ are calculated by diagonalizing
the matrix.

The layer magnetization of the layer $n$ is given by (see
technical details in Ref. 36):

$$\langle S_n^z \rangle = \frac{1}{2} - \frac{1}{\Delta} \int \int dk_x dk_z \sum_{i=1}^{2N} \frac{Q_{2n-1}(E_i)}{e^{E_i/k_BT} - 1}$$

(35)

where $n = 1, \ldots, N$, and $Q_{2n-1}(E_i)$ is the determinant
obtained by replacing the $(2n - 1)$-th column of $M$ by $u$
at $E_i$.

The layer magnetizations can be calculated at finite
temperatures self-consistently using the above formula.

The numerical method to carry out this task has been
described in Refs. 32. One can summarize

(i) the iteration is stopped. The method is thus
are the same as the inputs within a precision (usually at
(iii) if the output values

The value of the spin in the layer

$$‖S‖ = \frac{1}{2} + \int \int dk_x dk_z \sum_{i=1}^{N} Q_{2n-1}(E_i)$$

(36)

where the sum is performed over $N$ negative energies of $E_i$
for positive values, the Bose-Einstein factor in Eq. (35)
is equal to 0 at $T = 0$.

The transition temperature $T_c$ can be calculated by
letting $\langle S_n^z \rangle$ on the left-hand side of Eq. (35) to go to
zero. The energy $E_i$ tends then to zero, so that we can make
an expansion of the exponential at $T = T_c$. We have

$$\left(\frac{1}{k_BT_c}\right) = \frac{2}{\Delta} \int \int dk_x dk_z \sum_{i=1}^{2N} \frac{Q_{2n-1}(E_i)}{E_i}$$

(37)

IV. TWO AND THREE DIMENSIONS:
SPIN-WAVE SPECTRUM AND
MAGNETIZATION

Consider just one single $xz$ plane. The above matrix
is reduced to two coupled equations

$$(E + A_n)g_{n,n′} + B_n f_{n,n′} = 2 < S_n^z > \delta(n,n′)$$

$-B_n g_{n,n′} + (E - A_n) f_{n,n′} = 0$$

(38)

where $A_n$ is given by (32) but without $J_\perp$ term for the 2D
case considered here. Coefficients $B_n$ and $C_n$ are given
by (33) and (34) with $C_n = 0$. The poles of the GF are
the eigenvalues of the SW spectrum which are given by the
secular equation

$$(E + A_n)(E - A_n) + B_n^2 = 0$$

$$(E + A_n)(E - A_n) + B_n^2 = 0$$

$$E^2 - A_n^2 + B_n^2 = 0$$

$$E = \pm \sqrt{(A_n + B_n)[(A_n - B_n]}$$

(39)

where ± indicate the left and right SW precessions. Several
remarks are in order:

(i) if $\theta = 0$, we have $B_n = 0$ and the last three terms of
$A_n$ are zero. We recover then the ferromagnetic SW dispersion
relation

$$E = 2ZJ_n < S_n^z > (1 - \gamma)$$

(40)

where $Z = 4$ is the coordination number of the square
lattice (taking $d_n = 0$),

(ii) if $\theta = \pi$, we have $A_n = 8J_n < S_n^z >$, $B_n = -8J_n < S_n^z >$. We recover then the antiferromagnetic SW dispersion
relation

$$E = 2ZJ_n < S_n^z > \sqrt{1 - \gamma^2}$$

(41)

(iii) in the presence of a DM interaction, we have $0 < \cos \theta < 1$ ($0 < \theta < \pi/2$). If $d_n = 0$, the quantity in the
square root of Eq. (39) is always $\geq 0$ for any $\theta$. It is zero
at $\gamma = 1$. The SW spectrum is therefore stable at the
long-wavelength limit. The anisotropy $d_n$ gives a gap at $\gamma = 1$.

As said earlier, the necessity to include an anisotropy
does have a double purpose: it permits a gap and stabilizes a
long-range ordering at finite $T$ in 2D systems.

Figure 3 shows the SW spectrum calculated from Eq.
(39) for $\theta = 30$ degrees ($\pi/6$ radian) and 80 degrees (1.536
radian). The spectrum is symmetric for positive and neg-
ative wave vectors and for left and right precessions. Note
that for small $\theta$ (i.e. small $D$) $E$ is proportional to $k^2$ at
low $k$ (cf. Fig. 3a), as in ferromagnets. However, as $\theta$ in-
creases, we observe that $E$ becomes linear in $k$ as seen in
Fig. 3b. This is similar to antiferromagnets. The change of
behavior is progressive with increasing $\theta$; we do not
observe a sudden transition from $k^2$ to $k$ behavior. This
feature is also observed in three dimensions (3D) and in
thin films as seen below.

It is noted that, thanks to the existence of the
anisotropy $d$, we avoid the logarithmic divergence at
$k = 0$ so that we can observe a long-range ordering at
finite $T$ in 2D. We show in Fig. 4 the magnetization $M$
($\equiv < S^z >$) calculated by Eq. (35) for one layer using
$d = 0.001$. It is interesting to observe that $M$ depends
strongly on $\theta$: at high $T$, larger $\theta$ yields stronger $M$.
However, at $T = 0$ the spin length is smaller for larger
$\theta$ due to the so-called spin contraction [36] calculated by
We can use the Fourier transformation in the $y$ direction, namely $g_{n \pm 1} = g_n e^{\pm ik_y a}$ and $f_{n \pm 1} = f_n e^{\pm ik_y a}$. The matrix (30) is reduced to two coupled equations of $g$ and $f$ functions, omitting index $n$,
\[
(E + A')g + Bf = 2 < S^z >
-Bg + (E - A')f = 0
\]
where
\[
A' = -J_s[8 < S^z > \cos \theta (1 + d)
-4 < S^z > \gamma (\cos \theta + 1)]
+4J_s < S^z > \cos (k_y a)
-4D \sin \theta < S^z > \gamma
+8D \sin \theta < S^z >
\]
\[
B = 4J_s < S^z > \gamma (\cos \theta - 1)
-4D \sin \theta < S^z > \gamma
\]

The spectrum is given by
\[
E = \pm \sqrt{(A' + B)(A' - B)}
\]
If $\cos \theta = 1$ (ferromagnetic), one has $B = 0$. By regrouping the Fourier transforms in three directions, one obtains the 3D ferromagnetic dispersion relation $E = 2Z < S^z > (1 - \gamma^2)$ where $\gamma = (\cos (k_y a) + \cos (k_x a) + \cos (k_z a))/3$ and $Z = 6$, coordination number of the simple cubic lattice. Unlike the 2D case where the angle is inside the plane so that the antiferromagnetic case can be recovered by setting $\cos \theta = -1$ as seen above, one cannot use the above formula to find the antiferromagnetic case because in the 2D formulation it was supposed a ferromagnetic coupling between planes, namely there is no angle between adjacent planes in the above formulation.

The same consideration as in the 2D case treated above shows that for $d = 0$ the spectrum $E \geq 0$ for positive precession and $E \leq 0$ for negative precession, for any $\theta$.

The limit $E = 0$ is at $\gamma = 1$ ($k = 0$). Thus there is no instability due to the DM interaction. Using Eq. (45), we have calculated the 3D spectrum. This is shown in Fig. 5 for a small and a large value of $\theta$. As in the 2D case, we observe $E \propto k$ when $k \rightarrow 0$ for large $\theta$. Main properties of the system are dominated by the in-plane DM behavior.

Figure 6a displays the magnetization $M$ versus temperature $T$ for several values of $\theta$. As in the 2D case, when $\theta$ is not zero, the spins have a contraction at $T = 0$: a stronger $\theta$ yields a stronger contraction. This generates a magnetization cross-over at low $T$ shown in the inset of Fig. 6a. The spin length at $T = 0$ versus $\theta$ is displayed in Fig. 6b. Note that the spin contraction in 3D is smaller than that in 2D. This is expected since quantum fluctuations are stronger at lower dimensions.

V. THE CASE OF A THIN FILM: SPIN-WAVE SPECTRUM, LAYER MAGNETIZATIONS

In the 2D and 3D cases shown above, there is no need at $T = 0$ to use a small anisotropy $d$. However in the case of thin films shown below, due to the lack of neighbors at the surface, the introduction of a DM interaction
destabilizes the spectrum at long wave-length $\vec{k} = 0$. Depending on $\theta$, we have to use a value for $d_c$ larger or equal to a "critical value" $d_c$ to avoid imaginary SW energies at $\vec{k} = 0$. The critical value $d_c$ is shown in Fig. 7 for a 4-layer film. Note that at the perpendicular configuration $\theta = \pi/2$, no SW excitation is possible: SW cannot propagate in a perpendicular spin configuration since the wave-vectors cannot be defined.

We show now a SW spectrum at a given thickness $N$. There are $2N$ energy values half of them are positive and the other half negative (left and right precessions): $E_i$ ($i = 1, \ldots, 2N$). Figure 8 shows the case of a film of 8 layers with $J_i = J_{11} \equiv 1$ for a weak and a strong value of $D$ (small and large $\theta$). As in the 2D and 2D cases, for strong $D$, $E$ is proportional to $k$ at small $k$ (cf. Fig. 8b). It is noted that this behavior concerns only the first mode. The upper modes remain in the $k^2$ behavior.
Figure 9 shows the layer magnetizations of the first four layers in a 8-layer film (the other half is symmetric) for several values of $\theta$. In each case, we see that the surface layer magnetization is smallest. This is a general effect of the lack of neighbors for surface spins even when there is no surface-localized SW as in the present simple-cubic lattice case [36].

Let us touch upon the surface effect in the present model. We know that for the simple cubic lattice, if the interactions are the same everywhere in the film, then there is no surface localized modes, and this is true with DM interaction (see spectrum in Fig. 8) and without DM interaction (see Ref. 37). In order to create surface modes, we have to take the surface exchange interactions different from the bulk ones. Low-lying branches of surface modes which are "detached" from the bulk spectrum are seen in the SW spectrum shown in Fig. 11a with $J_s^\parallel = 0.5, J_s^\perp = 0.5$. These surface modes strongly affect the surface magnetization as observed in Fig. 11b: the surface magnetization is strongly diminished with increasing $T$. The role of surface-localized modes on the strong decrease of the surface magnetization as $T$ increases has already been analyzed more than 30 years ago [37].
VI. CONCLUDING REMARKS

By a self-consistent Green’s function theory, we obtain the expression of the spin-wave dispersion relation in 2D and 3D as well as in a thin film. Due to the competition between ferromagnetic interaction $J$ and the perpendicular DM interaction $D$, the GS is non-linear with an angle $\theta$ which is shown to explicitly depend on the ratio $D/J$. The spectrum is shown to depend on $\theta$ and the layer magnetization is calculated self-consistently as a function of temperature up to the critical temperature $T_c$.

We have obtained new and interesting results. In particular we have showed that (i) the spin-wave excitation in 2D and 3D crystals is stable at $T = 0$ with the non-collinear spin configuration induced by the DM interaction $D$ without the need of an anisotropy, (ii) in the case of thin films, we need a small anisotropy $d$ to stabilize the spin-wave excitations because of the lack of neighbors at the surface, (iii) the spin-wave energy $E$ depends on $D$, namely on $\theta$: at the long wave-length limit, $E$ is proportional to $k^2$ for small $D$ but $E$ is linear in $k$ for strong $D$, in 2D and 3D as well as in a thin film, (iv) quantum fluctuations are inhomogeneous for layer magnetizations near the surface, (v) unlike in some previous works, spin waves in systems with asymmetric DM interactions are found to be symmetric with respect to opposite propagation directions.

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FIG. 12. 12-layer film: (a) Layer magnetizations versus $T$ for $\theta = \pi/6$ and anisotropy $d = 0.1$. Red circles, blue squares, green void squares magenta circles, void turquoise triangles and brown triangles correspond respectively to first, second, third, fourth, fifth and sixth layer, (b) Gap at $k = \theta$ as a function of film thickness $N$ for $\theta = \pi/6$, $d = 0.1$ at $T = 0.1$. (c) Critical temperature $T_c$ versus the film thickness $N$ calculated with $\theta = \pi/6$ and $d = 0.1$ using Eq. (37). Note that for infinite thickness (namely 3D), $T_c \simeq 2.8$ for $d = 0.1$.


